arXiv:math/0404420v2 [math.AP] 8 Apr 2005

GLOBAL EXISTENCE FOR DIRICHLET-WAVE EQUATIONS WITH QUADRATIC NONLINEARITIES IN HIGH DIMENSIONS

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1. **Introduction.**

In this paper, we provide a proof of global existence of solutions to quasilinear wave equations with quadratic nonlinearities exterior to nontrapping obstacles. Specifically, let $\mathcal{K}$ be a compact, nontrapping obstacle with smooth boundary. We will then be looking for solutions to

$$\begin{cases} \Box u = Q(du, d^2u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K} \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g \\ u(t,\,\cdot\,)|_{\partial\mathcal{K}} = 0 \end{cases} \tag{1.1}$$

exterior to $\mathcal{K}$. Here

$$\Box = (\Box_{c_1}, \Box_{c_2}, \dots, \Box_{c_D})$$

is a vector-valued d'Alembertian with

$$\Box_{c_I} = \partial_t^2 - c_I^2 \Delta$$

and $c_I > 0$ for $I = 1, 2, \dots, D$.

Letting $\partial_0 = \partial_t$ when convenient, we can expand our quadratic, quasilinear forcing term Q as follows

$$Q^I(du, d^2u) = \sum_{\substack{0\le j,k\le n\\ 1\le J,K\le D}} A^{I,jk}_{JK}\partial_j u^J \partial_k u^K + \sum_{\substack{0\le j,k,l\le n\\ 1\le J,K\le D}} B^{IJ,jk}_{K,l}\partial_l u^K \partial_j\partial_k u^J, \quad 1 \le I \le D. \tag{1.2}$$

In order that we might apply the local existence results of Keel-Smith-Sogge [9] and in order to help guarantee hyperbolicity, we assume the following symmetry condition

$$B^{IJ,jk}_{K,l} = B^{JI,jk}_{K,l} = B^{IJ,kj}_{K,l}. \tag{1.3}$$

To solve (1.1), one must assume that the Cauchy data (f, g) satisfy certain compatibility conditions. Such conditions are well-known, and for further detail, we refer the reader to [9]. Briefly, if we let $J_k u = \{\partial_x^\alpha u \,:\, 0 \le |\alpha| \le k\}$, we can write $\partial_t^k u(0,\,\cdot\,) = \psi_k(J_k f, J_{k-1} g)$, $0 \le k \le m$, where u is any formal H^m solution to (1.1) and m is fixed. The ψ_k are called compatibility functions and depend on Q, $J_k f$, and $J_{k-1} g$. The compatibility condition for (1.1) with $(f, g) \in H^m \times H^{m-1}$ requires that ψ_k vanish on $\partial\mathcal{K}$ when $0 \le k \le m-1$. Additionally, we say that $(f, g) \in C^\infty$ satisfy the compatibility condition to infinite order if the above condition holds for all m.

The authors were supported in part by the NSF.

The main result of this paper asserts that such systems of multiple speed, Dirichlet-wave equations admit global solutions.

Theorem 1.1. *Assume $n \geq 4$, and let $\mathcal{K}$, Q and $\Box$ be as above. Suppose further that $(f, g) \in C^\infty(\mathbb{R}^n \backslash \mathcal{K})$ satisfy the compatibility conditions to infinite order. Then, there is a constant $\varepsilon_0 > 0$ and an integer $N > 0$ so that for all $\varepsilon \leq \varepsilon_0$, if*

$$\sum_{|\alpha| \leq N} \|\langle x\rangle^{|\alpha|} \partial_x^\alpha f\|_2 + \sum_{|\alpha| \leq N-1} \|\langle x\rangle^{1+|\alpha|} \partial_x^\alpha g\|_2 \leq \varepsilon, \tag{1.4}$$

then (1.1) *has a unique global solution $u \in C^\infty([0,\infty) \times \mathbb{R}^n \backslash \mathcal{K})$.*

Additionally, we note that the proof of the theorem would allow any forcing term $F(du, d^2u)$ vanishing to second order and linear in d^2u.

Global existence of solutions to boundaryless wave equations of the form (1.1) was first shown by Hörmander and Klainerman (see, e.g., [31]). A recent paper of Hidano [3] explores an alternate method of proof that admits the multiple speed setting.

In the obstacle setting, (1.1) was first considered by Shibata-Tsutsumi [26] and was shown to have global existence in spatial dimensions $n \geq 6$. Hayashi [2] was able to prove global existence exterior to a ball in all spatial dimensions $n \geq 4$. A result similar to Theorem 1.1 was shown by the first author [17] for semilinear equations.

In the case of $n = 3$, solutions to (1.1) exterior to certain obstacles were studied by Keel-Smith-Sogge [10, 11], the authors [19], and Metcalfe-Nakamura-Sogge [20]. As in these works, we will be using the exterior domain analog of Klainerman's method of commuting vector fields [12] as developed by Keel-Smith-Sogge [11]. In particular, we restrict our attention to the invariant vector fields that are admissible for the obstacle setting, $\{L, Z\}$, where Z represents the generators of the space-time translations and spatial rotations

$$Z = \{\partial_i, x_j\partial_k - x_k\partial_j\}, \quad 0 \leq i \leq n, \quad 1 \leq j, k \leq n$$

and where L is the scaling vector field

$$L = t\partial_t + r\partial_r.$$

Here and in what follows, $r = |x|$. We also set

$$\Omega = \{x_j\partial_k - x_k\partial_j\}, \quad 1 \leq j, k \leq n$$

to be the set of generators of spatial rotations.

The main new approach in this paper versus [19] is the techniques used to handle the boundary terms that necessarily arise when studying obstacle problems. In [19], these were handled using Huygens' principle. In the current setting, we develop simple local bounds for solutions to the Minkowski wave equation using the fundamental solution. We then use local energy decay and techniques of Smith-Sogge [30] to reduce to this case.

Also, as in [11], we will be using a class of weighted $L^2_tL^2_x$ estimates where the weight is a negative power of $\langle x\rangle = \langle r\rangle = \sqrt{1+r^2}$. Such estimates allow one to take advantage of the decay in $|x|$ which is much easier to prove in the obstacle setting than the more traditional decay in t. These estimates were first developed for even spatial dimensions by the first author in [17]. The proof relied on a local Minkowski version developed by

Smith-Sogge [30] and on other weighted estimates in [18]. Local versions of these weighted $L^2_tL^2_x$ estimates were originated in the obstacle setting using different techniques by Burq [1]. Burq's estimates relied on rather weak hypotheses, namely the existence of certain resolvant bounds. Since these resolvant bounds are implied by the local energy decay that we discuss next, we will assume Burq's bounds when convenient.

By a simple scaling argument, we may and will assume throughout that

$$\mathcal{K}\subset\{|x|\le 1\}.$$

The nontrapping assumption on the geometry of the obstacle, which states that there is a T_R such that no geodesic of length T_R is completely contained in $\{|x|\le R\}\cap\mathbb{R}^n\backslash\mathcal{K}$, allows us to refer to well-known local energy decay estimates. In particular, if u is a solution to the homogeneous wave equation

$$\begin{cases}(\partial_t^2-\Delta)u(t,x)=0\\ u(0,\,\cdot\,)=f,\quad \partial_t u(0,\,\cdot\,)=g\\ u(t,x)=0,\quad x\in\partial\mathcal{K}\end{cases}\tag{1.5}$$

and if the Cauchy data f,g are assumed to vanish for $|x|>4$, then there is a constant $c>0$ so that

$$\Bigl(\int_{\{x\in\mathbb{R}^n\backslash\mathcal{K}:|x|<4\}}|u'(t,x)|^2\,dx\Bigr)^{1/2}\le Ce^{-ct}\Bigl(\|\nabla_x f\|_2+\|g\|_2\Bigr)\tag{1.6}$$

if n is odd. Here $'=(\partial_t,\nabla_x)$ is the space-time gradient. We refer the reader to Taylor [34], Lax-Phillips [15], Vainberg [36], Morawetz-Ralston-Strauss [23], Strauss [33], and Morawetz [22].

In even spatial dimensions n, we have the weaker decay

$$\Bigl(\int_{\{x\in\mathbb{R}^n\backslash\mathcal{K}:|x|<4\}}|u'(t,x)|^2\,dx\Bigr)^{1/2}\le Ct^{-(n-1)}\Bigl(\|\nabla_x f\|_2+\|g\|_2\Bigr).\tag{1.7}$$

See Ralston [25]. We also refer the reader to Melrose [16] and Strauss [33]. We will not require the additional decay (1.6) and will only use (1.7) throughout.

One of the advantages of the proof that we shall use is that the argument can easily be altered to allow for the necessary loss of regularity in the right sides of (1.6) and (1.7) if the exterior domain contains trapped rays. The necessity of such a loss was shown by Ralston [24], and in $n=3$, Ikawa [7, 8] was able to show a version of (1.6) with a loss of regularity for certain exterior domains that contain hyperbolic trpaped rays. In [19, 20], considerations were taken to establish existence results in the presence of such geometries.

This paper is organized as follows. In the next section, we collect the L^2 energy estimates that we will require. These are $n\ge 4$ analogs of those developed by the authors in [19], and the proofs of these results extend trivially to the more general setting. In §3, we prove the necessary weighted $L^2_tL^2_x$ estimates. As mentioned previously, these follow easily from the estimates in [1] and [17] and are higher dimensional analogs of the estimates of Keel-Smith-Sogge [10, 11]. In §4, we state a few Sobolev-type results. These are exterior domain analogs of results proven and used by Klainerman [12], Klainerman-Sideris [13], Sideris [27], Sideris-Tu [29], and Hidano-Yokoyama [4, 5]. The extension of these estimates to the exterior domain follows exactly as in [20]. In §5, we provide proofs

of the estimates for the boundary terms. Finallly, in §6, we set up a continuity argument and use these estimates to prove global existence.

2. **Energy Type Estimates.**

In this section, we collect the energy estimates that we shall require. Unless stated otherwise, the proofs of these estimates can be found in [19] for the $n = 3$ case. These arguments, however, extend to general spatial dimensions $n \geq 2$ trivially.

Specifically, we will be concerned with solutions $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K})$ of the Dirichlet-wave equation

$$\begin{cases} \Box_\gamma u = F \\ u|_{\partial\mathcal{K}} = 0 \\ u|_{t=0} = f, \quad \partial_t u|_{t=0} = g \end{cases} \tag{2.1}$$

where

$$(\Box_\gamma u)^I = (\partial_t^2 - c_I^2 \Delta) u^I + \sum_{J=1}^D \sum_{j,k=0}^n \gamma^{IJ,jk}(t,x) \partial_j \partial_k u^J, \quad 1 \leq I \leq D.$$

We shall assume that the $\gamma^{IJ,jk}$ satisfy the symmetry conditions

$$\gamma^{IJ,jk} = \gamma^{JI,jk} = \gamma^{IJ,kj} \tag{2.2}$$

as well as the size condition

$$\sum_{I,J=1}^D \sum_{j,k=0}^n \|\gamma^{IJ,jk}(t,x)\|_\infty \leq \delta \tag{2.3}$$

for δ sufficiently small (depending on the wave speeds). The energy estimate will involve bounds for the gradient of the perturbation terms

$$\|\gamma'(t,\,\cdot\,)\|_\infty = \sum_{I,J=1}^D \sum_{j,k,l=0}^n \|\partial_l \gamma^{IJ,jk}(t,\,\cdot\,)\|_\infty,$$

and the energy form associated with $\Box_\gamma$, $e_0(u) = \sum_{I=1}^D e_0^I(u)$, where

$$\begin{aligned} e_0^I(u) = (\partial_0 u^I)^2 &+ \sum_{k=1}^n c_I^2 (\partial_k u^I)^2 \\ &+ 2 \sum_{J=1}^D \sum_{k=0}^n \gamma^{IJ,0k} \partial_0 u^I \partial_k u^J - \sum_{J=1}^D \sum_{j,k=0}^n \gamma^{IJ,jk} \partial_j u^I \partial_k u^J. \end{aligned} \tag{2.4}$$

The most basic estimate will lead to a bound for

$$E_M(t) = E_M(u)(t) = \int \sum_{j=0}^M e_0(\partial_t^j u)(t,x)\, dx.$$

Lemma 2.1. *Fix $M = 0, 1, 2, \dots,$ and assume that the perturbation terms $\gamma^{IJ,jk}$ are as above. Suppose also that $u \in C^\infty$ solves (2.1) and for every t, $u(t,x) = 0$ for large x. Then there is an absolute constant C so that*

$$\partial_t E_M^{1/2}(t) \le C \sum_{j=0}^{M} \|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\|_2 + C\|\gamma'(t,\,\cdot\,)\|_\infty E_M^{1/2}(t). \tag{2.5}$$

Before stating the next result, let us introduce some notation. If $P = P(t, x, D_t, D_x)$ is a differential operator, we shall let

$$[P, \gamma^{kl}\partial_k\partial_l]u = \sum_{1\le I,J\le D}\ \sum_{0\le k,l\le n} |[P, \gamma^{IJ,kl}\partial_k\partial_l]u^J|.$$

In order to allow the above energy estimate to include the more general vector fields L, Z, we will need to use a variant of the scaling vector field L. We fix a bump function $\eta \in C^\infty(\mathbb{R}^n)$ with $\eta(x) = 0$ for $x \in \mathcal{K}$ and $\eta(x) = 1$ for $|x| > 1$. Then, set $\tilde{L} = \eta(x) r\partial_r + t\partial_t$. Using this variant of the scaling vector field and an elliptic regularity argument, one can establish

Proposition 2.2. *Suppose that the constant in (2.3) is small. Suppose further that*

$$\|\gamma'(t,\,\cdot\,)\|_\infty \le \delta/(1+t), \tag{2.6}$$

and

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \Big(\|\tilde{L}^\mu \partial_t^j \Box_\gamma u(t,\,\cdot\,)\|_2 + \|[\tilde{L}^\mu\partial_t^j, \gamma^{kl}\partial_k\partial_l]u(t,\,\cdot\,)\|_2 \Big) \le \frac{\delta}{1+t} \sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\tilde{L}^\mu \partial_t^j u'(t,\,\cdot\,)\|_2 + H_{\nu_0,N_0}(t), \tag{2.7}$$

where N_0 and ν_0 are fixed. Then

$$\begin{aligned}
\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} &\|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \\
\le C &\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha \Box u(t,\,\cdot\,)\|_2 + C(1+t)^{A\delta} \sum_{\substack{\mu+j\le N_0+\nu_0\\ \mu\le\nu_0}} \Big(\int e_0(\tilde{L}^\mu \partial_t^j u)(0,x)\,dx\Big)^{1/2} \\
&+ C(1+t)^{A\delta}\Big(\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0-1}} \|L^\mu \partial^\alpha \Box u(s,\,\cdot\,)\|_2\,ds + \int_0^t H_{\nu_0,N_0}(s)\,ds\Big) \\
&+ C(1+t)^{A\delta} \int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0-1}} \|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds,
\end{aligned} \tag{2.8}$$

where the constants C and A are absolute constants.

In practice $H_{\nu_0,N_0}(t)$ will involve weighted L^2_x norms of $|L^\mu\partial^\alpha u'|^2$ with $\mu+|\alpha|$ much smaller than $N_0+\nu_0$, and so the integral involving H_{ν_0,N_0} can be dealt with using an inductive argument and the weighted $L^2_tL^2_x$ estimates of the subsequent section.

In proving our existence results for (1.1), a key step will be to obtain a priori L^2-estimates involving $L^\mu Z^\alpha u'$. Begin by setting

$$Y_{N_0,\nu_0}(t)=\int\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} e_0(L^\mu Z^\alpha u)(t,x)\,dx. \tag{2.9}$$

We, then, have the following proposition which shows how the $L^\mu Z^\alpha u'$ estimates can be obtained from the ones involving $L^\mu\partial^\alpha u'$.

Proposition 2.3. *Suppose that the constant δ in* (2.3) *is small and that* (2.6) *holds. Then,*

$$\begin{aligned}\partial_t Y_{N_0,\nu_0}\le CY^{1/2}_{N_0,\nu_0}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|\Box_\gamma L^\mu Z^\alpha u(t,\,\cdot\,)\|_2+C\|\gamma'(t,\,\cdot\,)\|_\infty Y_{N_0,\nu_0}\\ +C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|^2_{L^2(|x|<1)}.\end{aligned} \tag{2.10}$$

3. $\mathbf{L^2_tL^2_x}$ Estimates.

As in Keel-Smith-Sogge [10, 11], we will require a class of weighted $L^2_tL^2_x$ estimates. They will be used, for example, to control the local L^2 norms such as the last term in (2.10). For convenience, allow $\Box=\partial_t^2-\Delta$ to denote the unit speed, scalar d'Alembertian for the remainder of the section. The transition to the general case is straightforward. Also, set

$$S_T=\{[0,T]\times\mathbb{R}^n\backslash\mathcal{K}\}$$

to be the time strip of height T in $\mathbb{R}_+\times\mathbb{R}^n\backslash\mathcal{K}$. Here we will study solutions of the wave equation with vanishing initial data. In the sequel, we will reduce to this case.

We first note that if u is a solution to

$$\begin{cases}\Box u(t,x)=F(t,x)+G(t,x), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n\backslash\mathcal{K}\\ u(t,x)|_{\partial\mathcal{K}}=0\\ u(t,x)=0, & t<0\end{cases} \tag{3.1}$$

where $G(t,x)=0$ for $|x|>2$, then we have that

$$\|u'\|_{L^2_tL^2_x([0,t]\times\{|x|<2\})}\le C\int_0^t\|F(s,\,\cdot\,)\|_2\,ds+C\|G\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n\backslash\mathcal{K})}. \tag{3.2}$$

Indeed, (3.2) was shown to follow from certain resolvant estimates in Burq [1] (Theorem 3). Since the local energy decay estimates (1.6) and (1.7) imply these resolvant estimates, we assume (3.2).

Since $[\partial_t,\Box]=0$ and since ∂_t preserves the boundary condition, (3.2) holds with u replaced by ∂_t^ju and F,G replaced by $\partial_t^jF,\partial_t^jG$ respectively for any $j=0,1,2,\dots$. By

elliptic regularity (see Lemma 2.3 of [19]), it follows that

$$\sum_{|\alpha|\le N}\|\partial^\alpha u'\|_{L^2_tL^2_x([0,t]\times\{|x|<2\})}\le C\sum_{|\alpha|\le N}\int_0^t\|\partial^\alpha F(s,\,\cdot\,)\|_2\,ds \\ +C\sum_{|\alpha|\le N-1}\|\partial^\alpha F\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n\backslash\mathcal{K})}+C\sum_{|\alpha|\le N}\|\partial^\alpha G\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n\backslash\mathcal{K})} \tag{3.3}$$

if G is as above. Moreover, using an inductive argument in ν_0, we can prove

Lemma 3.1. *Suppose $n\ge 3$, and suppose that $\mathcal{K}$ is nontrapping. Let u be a solution to (3.1), and suppose $G(t,x)=0$ for $|x|>2$. Then, for any integers $\nu_0, N\ge 0$, we have*

$$\sum_{\substack{|\alpha|+\mu\le N+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha u'\|_{L^2_tL^2_x([0,t]\times\{|x|<2\})}\le C\sum_{\substack{|\alpha|+\mu\le N+\nu_0\\ \mu\le\nu_0}}\int_0^t\|L^\mu\partial^\alpha F(s,\,\cdot\,)\|_2\,ds \\ +C\sum_{\substack{|\alpha|+\mu\le N+\nu_0-1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha F\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n\backslash\mathcal{K})}+C\sum_{\substack{|\alpha|+\mu\le N+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha G\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n\backslash\mathcal{K})}. \tag{3.4}$$

Proof of Lemma 3.1: We will indeed use induction on ν_0 where (3.3) serves as the base case $\nu_0=0$. We now proceed under the the assumption that (3.4) holds for any N with ν_0 replaced by ν_0-1.

Letting $\tilde{L}$ be as in the previous section, we see that the left side of (3.4) is dominated by

$$\sum_{|\alpha|\le N}\|L^{\nu_0-1}\partial^\alpha(\tilde{L}u)'\|_{L^2_tL^2_x([0,t]\times\{|x|<2\})}+\sum_{\substack{|\alpha|+\mu\le N+\nu_0\\ \mu\le\nu_0-1}}\|L^\mu\partial^\alpha u'\|_{L^2_tL^2_x([0,t]\times\{|x|<2\})}.$$

By the inductive hypothesis, the second term is trivially controlled by the right side of (3.4).

For the first term, we note that

$$\Box(\tilde{L}u)=\tilde{L}\Box u+[\Box,\tilde{L}]u=\tilde{L}\Box u+2\Box u+[\Box,(1-\eta(x))r\partial_r]u \\ =\Big(\tilde{L}\Box u+2\Box u\Big)+\Big(-(\Delta\eta)r\partial_r u-2\partial_r\eta\partial_r u-2r\nabla\eta\,\cdot\,\nabla_x(\partial_r u)+2(1-\eta)\partial_r^2u\Big). \tag{3.5}$$

Notice, in particular, that the second grouping of terms are all supported in $|x|<1$. Thus, if we apply the inductive hypothesis to $\tilde{L}u$, it follows that the left side of (3.4) is bounded by the right side of (3.4) plus

$$\sum_{\substack{|\alpha|+\mu\le N+\nu_0\\ \mu\le\nu_0-1}}\|L^\mu\partial^\alpha u'\|_{L^2_tL^2_x([0,t]\times\{|x|<1\})}.$$

This last term, using the inductive hypothesis, is also easily seen to be controlled by the right side of (3.4) which completes the proof. □

We will also require the associated global results in Minkowski space. Let v be a solution to the boundaryless wave equation

$$\begin{cases} \Box v(t,x) = F(t,x) + G(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ v(t,x) = 0, \quad t < 0. \end{cases} \tag{3.6}$$

We, then, have the following result of the first author [17] (Proposition 2.2 and Proposition 2.7)

Lemma 3.2. *Suppose $n \geq 4$, and let v be a solution to (3.6). If $G(s,x) = 0$ for $|x| > 2$, then*

(3.7)

$$\sum_{\substack{|\alpha|+\mu\leq N+\nu \\ \mu\leq\nu}} \|\langle r\rangle^{-(n-1)/4} L^\mu Z^\alpha v'\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n)} \leq C\int_0^t \sum_{\substack{|\alpha|+\mu\leq N+\nu \\ \mu\leq\nu}} \|L^\mu Z^\alpha F(s,\,\cdot\,)\|_2\,ds + C\sum_{\substack{|\alpha|+\mu\leq N+\nu \\ \mu\leq\nu}} \|L^\mu \partial^\alpha G\|_{L^2_tL^2_x([0,t]\times\mathbb{R}^n)}$$

for any $N, \nu \geq 0$.

A global estimate for the Dirichlet-wave equation will follow from the local estimate (3.4) and the global Minkowski estimate (3.7). In particular, we have the following $n \geq 4$ analog of Theorem 6.3 of Keel-Smith-Sogge [11].

Proposition 3.3. *Fix N_0 and ν_0. Suppose that $\mathcal{K}$ is nontrapping. Suppose, also, that $u \in C^\infty$, $u|_{\partial\mathcal{K}} = 0$, and $u(t,x) = 0$ for $t < 0$. Then, there is a constant $C = C_{N_0,\nu_0,\mathcal{K}}$ so that if u vanishes for large x for every fixed t,*

(3.8)

$$\sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \|\langle x\rangle^{-(n-1)/4} L^\mu \partial^\alpha u'\|_{L^2_tL^2_x(S_T)} \leq C\int_0^T \sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \|\Box L^\mu \partial^\alpha u(s,\,\cdot\,)\|_2\,ds + C\sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0-1 \\ \mu\leq\nu_0}} \|\Box L^\mu \partial^\alpha u\|_{L^2_tL^2_x(S_T)}.$$

Additionally,

(3.9)

$$\sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \|\langle x\rangle^{-(n-1)/4} L^\mu Z^\alpha u'\|_{L^2_tL^2_x(S_T)} \leq C\int_0^T \sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \|\Box L^\mu Z^\alpha u(s,\,\cdot\,)\|_2\,ds + C\sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0-1 \\ \mu\leq\nu_0}} \|\Box L^\mu Z^\alpha u\|_{L^2_tL^2_x(S_T)}.$$

While the above proposition is stated for nontrapping geometries, the same argument will yield estimates for any geometry satisfying the resolvant bounds used in [1] provided that a sufficient loss of regularity is allowed for in the right sides. See [19] (Proposition

2.6) for an $n = 3$ example. Additionally, we note that (3.7), (3.8), and (3.9) hold with the weight $\langle x\rangle^{-(n-1)/4}$ in the left replaced by $\langle x\rangle^{-1/2-\varepsilon}$ for any $\varepsilon > 0$. We refer the reader to the scaling argument in Keel-Smith-Sogge [10] (Proposition 2.1) and to the application of such estimates in Metcalfe-Sogge-Stewart [21] (Proposition 2.3). In the sequel, as in [17], we will only require the estimates as stated.

Proof of Proposition 3.3: Let us prove only (3.8) as (3.9) follows from the same arguments.

Since the better estimates (3.4) hold when the norm in the left is taken over $S_T \cap \{|x| < 2\}$, it suffices to consider the norm in the left over $S_T \cap \{|x| \ge 2\}$. To do this, we fix $\beta \in C^\infty(\mathbb{R}^n)$ satisfying $\beta(x) \equiv 1$, $|x| \ge 2$ and $\beta(x) \equiv 0$, $|x| < 3/2$. Since we are assuming $\mathcal{K} \subset \{|x| < 1\}$, it follows that $v = \beta u$, which is equal to u over $|x| \ge 2$, solves the Minkowski wave equation

$$\Box v = \beta \Box u - 2\nabla\beta \cdot \nabla_x u - (\Delta\beta)u$$

with vanishing initial data. Here, we apply (3.7) with F replaced by $\beta\Box u$ and G replaced by $-2\nabla\beta\cdot\nabla_x u - (\Delta\beta)u$. It is essential to note that G vanishes unless $|x| < 2$. Thus, by (3.9), we have that

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|\langle x\rangle^{-(n-1)/4} L^\mu \partial^\alpha v'\|_{L^2_tL^2_x(S_T)} \le C\int_0^T \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2\, ds$$
$$+ C \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u'\|_{L^2_tL^2_x(S_T\cap\{|x|<2\})}.$$

Here, we have used the fact that the Dirichlet boundary condition allows us to control u locally by u'. The bound for the last term on the right follows from (3.4), which completes the proof. $\Box$

4. **Sobolev-type Estimates.**

In the sequel, we will require a number of Sobolev-type estimates. These are useful for establishing pointwise decay estimates that we be required in the continuity argument.

We begin with a now standard weighted Sobolev estimate (see [12]).

Lemma 4.1. *Suppose that $h \in C^\infty(\mathbb{R}^n)$. Then, for $R \ge 1$,*

$$\|h\|_{L^\infty(R/2<|x|<R)} \le CR^{-(n-1)/2} \sum_{|\alpha|+|\beta|\le(n+2)/2} \|\Omega^\alpha\partial_x^\beta h\|_{L^2(R/4<|x|<2R)}, \tag{4.1}$$

and

$$\|h\|_{L^\infty(R-1<|x|<R)} \le CR^{-(n-1)} \sum_{|\alpha|+|\beta|\le n} \|\Omega^\alpha\partial_x^\beta h\|_{L^1(R-2<|x|<R+1)}. \tag{4.2}$$

Next, we will need the following estimates for the boundaryless case. The first is due to Klainerman-Sideris [13] and says that if $g \in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^n)$, then

$$\|\langle t-r\rangle\partial^2 g(t,\,\cdot\,)\|_2 \le C\sum_{|\alpha|\le 1}\|\Gamma^\alpha g'(t,\,\cdot\,)\|_2 + C\|\langle t+r\rangle(\partial_t^2-\Delta)g(t,\,\cdot\,)\|_2 \tag{4.3}$$

where $\Gamma = \{L, Z\}$. This was shown in [13] for the $n = 3$ case, but the proof is clearly valid for any $n \geq 2$. We also have the related estimate

$$r^{(n/2)-1}\langle t-r\rangle|\partial g(t,x)| \leq C \sum_{|\alpha|\leq n/2} \|Z^\alpha \partial g(t,\,\cdot\,)\|_2 + C \sum_{|\alpha|\leq n/2} \|\langle t-r\rangle Z^\alpha \partial^2 u(t,\,\cdot\,)\|_2. \tag{4.4}$$

This bound was shown in Hidano [3]. It is a generalization of the $n = 3$ bound of [4]. The latter follows easily from an estimate of Sideris [27].

If we argue as in [20] (Lemma 4.2, Lemma 4.3), the above estimates can be extended to the exterior domain as follows.

Lemma 4.2. *Suppose that $u \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n\backslash\mathcal{K})$ vanishes for $x \in \partial\mathcal{K}$. Then, if $|\alpha| = M$ and ν are fixed,*

$$\begin{aligned}\|\langle t-r\rangle L^\nu Z^\alpha \partial^2 u(t,\,\cdot\,)\|_2 &\leq C \sum_{\substack{|\beta|+\mu\leq M+\nu+1\\ \mu\leq\nu+1}} \|L^\mu Z^\beta u'(t,\,\cdot\,)\|_2 \\ &+ C \sum_{\substack{|\beta|+\mu\leq M+\nu\\ \mu\leq\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\,\cdot\,)\|_2 + (1+t)\sum_{\mu\leq\nu}\|L^\mu u'(t,\,\cdot\,)\|_{L^2(|x|<2)}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}r^{(n/2)-1}\langle t-r\rangle|\partial L^\nu Z^\alpha u(t,x)| &\leq C \sum_{\substack{|\beta|+\mu\leq M+\nu+(n/2)+1\\ \mu\leq\nu+1}} \|L^\mu Z^\beta u'(t,\,\cdot\,)\|_2 \\ &+ C \sum_{\substack{|\beta|+\mu\leq M+\nu+(n/2)\\ \mu\leq\nu}} \|\langle t+r\rangle L^\mu Z^\beta(\partial_t^2-\Delta)u(t,\,\cdot\,)\|_2 + C(1+t)\sum_{\mu\leq\nu}\|L^\mu u'(t,\,\cdot\,)\|_{L^2(|x|<2)}.\end{aligned} \tag{4.6}$$

5. **Boundary Term Estimates.**

In the sequel, we will need to control boundary terms such as those that appear in (2.8), (4.5), and (4.6). We will first need a result for solutions to free wave equations.

Lemma 5.1. *Suppose $n \geq 4$, and suppose that $u_0 \in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^n)$ is a solution to the boundaryless wave equation $\Box u_0 = G$ with vanishing initial data. Then,*

$$\begin{aligned}\int_0^t \|u(s,\,\cdot\,)\|_{L^2(|x|<3)}\,ds &\leq C\int_0^t \|G(s,\,\cdot\,)\|_2\,ds \\ &+ C\int_0^t\int \sum_{|\alpha|+|\beta|\leq n} |\Omega^\alpha\partial_y^\beta G(s,y)|\,\frac{dy\,ds}{|y|^{(n-1)/2}}.\end{aligned} \tag{5.1}$$

Proof of Lemma 5.1: Using cutoffs, it suffices to consider the solution $u(s,\,\cdot\,)$ in three cases: (1) $G(\tau,y)$ is supported in $|y| < 10$, (2) $G(\tau,y)$ is supported in $||y|-(s-\tau)| < 10$, and (3) $G(\tau,y)$ vanishes unless $|y| > 8$ and $||y|-(s-\tau)| > 8$.

The first two cases are handled quite easily. In the first, we can use the local energy decay (1.7) to see that

$$\|u(s,\,\cdot\,)\|_{L^2(|x|<3)} \leq C\int_0^s \frac{1}{(1+s-\tau)^{n-1}}\|G(\tau,\,\cdot\,)\|_2\,d\tau.$$

For the second case, we have

$$\|u(s,\,\cdot\,)\|_{L^2(|x|<3)} \le C\|u(s,\,\cdot\,)\|_{L^{\frac{2n}{n-2}}(|x|<3)} \le C\int_0^s \|G(\tau,\,\cdot\,)\|_{L^2(||y|-(s-\tau)|<10)}\,d\tau$$

by Sobolev estimates and the energy inequality.

Thus, we only need to establish a bound in the third case. Here, we use the fact that

$$u(s,x) = \int_0^s R(s-\tau,\,\cdot\,) * G(\tau,\,\cdot\,)\,d\tau$$

where

$$R(t,x) = \lim_{\varepsilon\searrow 0} c_n' \mathrm{Im}(|x|^2 - (t-i\varepsilon)^2)^{-(n-1)/2}.$$

See, e.g., Taylor [35] p.222. From this, it clearly follows that

$$|u(s,x)| \le \int_0^s \int \frac{1}{((s-\tau)^2 - |x-y|^2)^{(n-1)/2}} |G(\tau,y)|\,dy\,d\tau.$$

By support considerations, the right side is bounded by

$$\int_0^s \int \frac{1}{\langle s-\tau-|y|\rangle^{(n-1)/2}} |G(\tau,y)|\,\frac{dy\,d\tau}{|y|^{(n-1)/2}}$$

when $|x|<3$.

It follows then that

$$\begin{aligned}\|u(s,\,\cdot\,)\|_{L^2(|x|<3)} \le C\int_0^s &\frac{1}{(1+s-\tau)^{n-1}}\|G(\tau,\,\cdot\,)\|_2\,d\tau \\ &+ C\int_0^s \|G(\tau,\,\cdot\,)\|_{L^2(||y|-(s-\tau)|<10)}\,d\tau \\ &\qquad + C\int_0^s\int \frac{1}{\langle s-\tau-|y|\rangle^{(n-1)/2}}|G(\tau,y)|\,\frac{dy\,d\tau}{|y|^{(n-1)/2}}.\end{aligned}$$

Thus, it is clear that upon integration, we have

$$\begin{aligned}\int_0^t \|u(s,\,\cdot\,)\|_{L^2(|x|<3)}\,ds \le C\int_0^t \|G(s,\,\cdot\,)\|_2\,ds \\ + C\int_0^t\int_0^s \|G(\tau,\,\cdot\,)\|_{L^2(||y|-(s-\tau)|<10)}\,d\tau\,ds + C\int_0^t\int |G(s,y)|\,\frac{dy\,ds}{|y|^{(n-1)/2}}\end{aligned} \tag{5.2}$$

provided $n \ge 4$.

The second term on the right side of (5.2) is dominated by

$$C\int_0^t\int_0^s (s-\tau)^{(n-1)/2}\|G(\tau,\,\cdot\,)\|_{L^\infty(||y|-(s-\tau)|<10)}\,d\tau\,ds.$$

By (4.2), this is in turn controlled by

$$C\int_0^t\int_0^s\int_{||y|-(s-\tau)|<20} \sum_{|\alpha|+|\beta|\le n} |\Omega^\alpha\partial^\beta G(\tau,y)|\,\frac{dy}{|y|^{(n-1)/2}}\,d\tau\,ds.$$

Since the sets $\{(\tau,y)\,:\,||y|-(j-\tau)|<20\}$, $j=0,1,2,\dots$ have finite overlap, we conclude that this is bounded by

$$C\int_0^t\int\sum_{|\alpha|+|\beta|\le n}|\Omega^\alpha\partial_x^\beta G(s,y)|\,\frac{dy\,ds}{|y|^{(n-1)/2}}.$$

With this bound, (5.1) follows immediately from (5.2). □

We now show how this yields our desired estimates for solutions to Dirichlet wave equations. The following is a generalized version of (2.32) in [19].

Lemma 5.2. *Suppose $n\ge 4$, and suppose that $u\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n\backslash\mathcal{K})$ vanishes for $x\in\partial\mathcal{K}$. Then, if N_0 and $\nu\le 1$ are fixed,*

(5.3)
$$\int_0^t\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds\le C\int_0^t\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2\,ds$$
$$+C\int_0^t\int\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+n+1\\ \mu\le\nu_0}}|L^\mu Z^\alpha\Box u(s,\,\cdot\,)|\,\frac{dy\,ds}{|y|^{(n-1)/2}}.$$

Proof of Lemma 5.2: Here, we examine two cases separately: (1) $\Box u(s,y)=0$ for $|y|\ge 4$, and (2) $\Box u(s,y)=0$ for $|y|\le 3$. For the former case, we have

$$\int_0^t\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<1)}\le C\int_0^t\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2\,ds. \tag{5.4}$$

Indeed, (1.7) yields

$$\sum_{\substack{j+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|\langle t\rangle^\mu\partial_t^\mu\partial_t^j u'(t,\,\cdot\,)\|_{L^2(|x|<1)}$$
$$\le C\int\frac{1}{(1+t-s)^{n-1-\nu_0}}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_{L^2(|x|<4)}\,ds.$$

Thus, by elliptic regularity (see Lemma 2.3 of [19]), it follows that

$$\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha u'(t,\,\cdot\,)\|_{L^2(|x|<1)}\le C\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0-1\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(t,\,\cdot\,)\|_2 \tag{5.5}$$
$$+C\int_0^t\frac{1}{(1+t-s)^{n-1-\nu_0}}\sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}}\|L^\mu\partial^\alpha\Box u(s,\,\cdot\,)\|_2\,ds.$$

This clearly implies (5.4) for $\nu_0\le 1$ and $n\ge 4$.

In the second case, the case that $\Box u$ vanishes near the obstacle, we write $u=u_0+u_r$ where u_0 solves the boundaryless wave equation $\Box u_0=\Box u$ with zero initial data. Fixing

$\beta \in C^\infty(\mathbb{R}^n)$ satisfying $\beta(x) \equiv 1$, $|x| < 2$, and $\beta(x) \equiv 0$ for $|x| > 3$, we set $\tilde{u} = \beta u_0 + u_r$. Clearly, $u = \tilde{u}$ for $|x| < 2$, and $\tilde{u}$ solves

$$\Box \tilde{u} = -2\nabla\beta \cdot \nabla_x u - (\Delta\beta)u_0$$

which is supported in $|x| < 3$. Thus, from (5.4), it follows that

$$\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0\\ \mu\le\nu_0}} \|L^\mu \partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds \le C\int_0^t \sum_{\substack{|\alpha|+\mu\le N_0+\nu_0+1\\ \mu\le\nu_0}} \|L^\mu\partial^\alpha u_0\|_{L^2(|x|<3)}\,ds.$$

Since $[\Box, L] = 2\Box$ and $[\Box, \partial] = 0$, (5.3) is a consequence of (5.1). $\Box$

6. **Proof of Theorem 1.1.**

In this section, we prove the global existence theorem, Theorem 1.1, when $n = 4$. Straightforward modifications will yield the general case $n \ge 4$. We take $N = 101$ in the smallness hypothesis (1.4); this, however, is not optimal.

The proof of global existence will rely on the following standard local existence theorem.

Theorem 6.1. *Suppose that f and g are as in Theorem 1.1 with $N \ge (3n+6)/2$ in* (1.4) *if n is even, $N \ge (3n+3)/2$ if n is odd. Then there is a $T > 0$ so that the initial value problem* (1.1) *with this initial data has a C^2 solution satisfying*

$$u \in L^\infty([0,T]; H^N(\mathbb{R}^n\backslash\mathcal{K})) \cap C^{0,1}([0,T]; H^{N-1}(\mathbb{R}^n\backslash\mathcal{K})).$$

The supremum of such T is equal to the supremum of all T such that the initial value problem has a C^2 solution with $\partial^\alpha u$ bounded for $|\alpha| \le 2$. Also, one can take $T \ge 2$ if $\|f\|_{H^N} + \|g\|_{H^{N-1}}$ is sufficiently small.

This esssentially follows from the local existence results Theorem 9.4 and Lemma 9.6 of Keel-Smith-Sogge [9]. The latter were only stated for diagonal single-speed systems; however, since the proof relied only on energy estimates, it extends to the multi-speed, non-diagonal case if the symmetry assumptions (1.3) are satisfied.

Next, in order to avoid dealing with difficulties involving the compatibility conditions for the Cauchy data, it is convenient to follow the example of Keel-Smith-Sogge [11] and reduce to an equivalent equation with vanishing initial data. We first note that if the initial data satisfy (1.4) with ε sufficiently small, then we can find a solution u to (1.1) on a set of the form $0 < ct < |x|$ where $c = 5\max_I c_I$, and this solution satisfies

$$\sup_{0<t<\infty} \sum_{|\alpha|\le 101} \|\langle x\rangle^{|\alpha|}\partial^\alpha u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n\backslash\mathcal{K}\,:\,|x|>ct)} \le C_0\varepsilon. \tag{6.1}$$

Rather than providing unnecessary technicalities, we refer the reader to [11], [19], or [20].

We will use this local solution u to allow us to restrict to the case of vanishing Cauchy data. To do so, we fix a cutoff function $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi(s) \equiv 1$ for $s \le \frac{1}{2c}$ and $\chi(s) \equiv 0$ for $s > \frac{1}{c}$, and we set

$$u_0(t,x) = \eta(t,x)u(t,x), \quad \eta(t,x) = \chi(|x|^{-1}t).$$

Assuming, as we may, that $0 \in \mathcal{K}$, we have that $|x|$ is bounded below on the complement of $\mathcal{K}$ and the function $\eta(t,x)$ is smooth and homogeneous of degree 0 in (t,x). Note that by (4.1) and (6.1), it follows that there is an absolute constant C_1 so that

$$(1+t+|x|)\sum_{\mu+|\alpha|\le 98}|L^\mu Z^\alpha u_0(t,x)| + \sum_{\mu+|\alpha|+|\beta|\le 101}\|\langle t+r\rangle^{|\beta|}L^\mu Z^\alpha \partial^\beta u_0(t,\,\cdot\,)\|_2 \le C_1\varepsilon. \tag{6.2}$$

Notice this also implies that

$$\sum_{\mu+|\alpha|+|\beta|\le 100}\|\langle x\rangle^{-3/4}L^\mu Z^\alpha u_0'\|_{L^2(S_t)}$$

is $O(\varepsilon)$.

Since

$$\Box u_0 = \eta Q(du, d^2u) + [\Box,\eta]u,$$

u solves $\Box u = Q(du,d^2u)$ for $0<t<T$ if and only if $w = u-u_0$ solves

$$\begin{cases}\Box w = (1-\eta)Q(du,d^2u) - [\Box,\eta]u\\ w|_{\partial\mathcal{K}} = 0\\ w(t,x) = 0, \quad t\le 0\end{cases} \tag{6.3}$$

for $0<t<T$.

If we let v be the solution of the linear equation

$$\begin{cases}\Box v = -[\Box,\eta]u\\ v|_{\partial\mathcal{K}} = 0\\ v(t,x) = 0, \quad t\le 0,\end{cases} \tag{6.4}$$

then we will show that (6.1) implies that there is another absolute constant C_2 so that

$$\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha v'(t,\,\cdot\,)\|_2 + \sum_{\mu+|\alpha|\le 100}\|\langle x\rangle^{-3/4}L^\mu Z^\alpha v'(t,\,\cdot\,)\|_{L^2_tL^2_x(S_t)} \le C_2\varepsilon. \tag{6.5}$$

Indeed, we can examine the first term in (6.5) using the standard energy integral method. Doing so, we see that

$$\begin{aligned}\partial_t \sum_{\mu+|\alpha|\le 99}&\|L^\mu Z^\alpha v'(t,\,\cdot\,)\|_2^2\\ &\le C\Big(\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha v'(t,\,\cdot\,)\|_2\Big)\Big(\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha \Box v(s,\,\cdot\,)\|_2\Big)\\ &\qquad + C\sum_{\mu+|\alpha|\le 99}\Big|\int_{\partial\mathcal{K}}\partial_0 L^\mu Z^\alpha v(t,\,\cdot\,)\nabla L^\mu Z^\alpha v(t,\,\cdot\,)\cdot n\, d\sigma\Big|,\end{aligned}$$

where n is the outward normal at a given point on $\partial\mathcal{K}$. Since $\mathcal{K}\subset\{|x|<1\}$ and since $\Box v=-[\Box,\eta]u$, it follows that the right side of the equation above is dominated by

$$C\Big(\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha v'(t,\,\cdot\,)\|_2\Big)\Big(\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha[\Box,\eta]u(s,\,\cdot\,)\|_2\Big) \\ + C\int_{\{x\in\mathbb{R}^4\backslash\mathcal{K}\,:\,|x|<1\}}\sum_{\mu+|\alpha|\le 100}|L^\mu Z^\alpha v'(t,x)|^2\,dx.$$

Integrating in t, this yields

$$\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha v'(t,\,\cdot\,)\|_2^2\le C\Big(\int_0^t\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha[\Box,\eta]u(s,\,\cdot\,)\|_2\,ds\Big)^2 \\ + C\int_0^t\sum_{\mu+|\alpha|\le 100}\|L^\mu Z^\alpha v'(s,\,\cdot\,)\|^2_{L^2(|x|<1)}\,ds. \tag{6.6}$$

The last term in (6.6) is dominated by the square of the second term in the left side of (6.5).

Thus, by (3.9), it follows that the square of the left side of (6.5) is controlled by

$$C\Big(\int_0^t\sum_{\mu+|\alpha|\le 100}\|L^\mu Z^\alpha[\Box,\eta]u(s,\,\cdot\,)\|_2\,ds\Big)^2+C\sum_{\mu+|\alpha|\le 99}\|L^\mu Z^\alpha[\Box,\eta]u\|^2_{L^2_tL^2_x(S_t)}.$$

Both of these terms are $O(\varepsilon^2)$ by (6.1), which completes the proof of (6.5).

Using this, we are now ready to set up the continuity argument which will complete the proof of Theorem 1.1. If $\varepsilon>0$ is as above, we shall assume that we have a C^2 solution of (6.3) for $0\le t\le T$ satisfying the lossless estimates

$$\sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 1}}\|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2+\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}}\|\langle x\rangle^{-3/4}L^\mu Z^\alpha w'\|_{L^2_tL^2_x(S_t)}\le A_0\varepsilon \tag{6.7}$$

$$(1+t+r)\sum_{|\alpha|\le 40}|Z^\alpha w'(t,x)|\le B_1\varepsilon, \tag{6.8}$$

as well as, the lossy higher order estimates

$$\sum_{|\alpha|\le 100}\|\partial^\alpha w'(t,\,\cdot\,)\|_2\le B_2\varepsilon(1+t)^{1/40} \tag{6.9}$$

$$\sum_{\substack{|\alpha|+\mu\le 81\\ \mu\le 2}}\|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2\le B_3\varepsilon(1+t)^{1/20} \tag{6.10}$$

$$\sum_{\substack{|\alpha|+\mu\le 79\\ \mu\le 2}}\|\langle x\rangle^{-3/4}L^\mu Z^\alpha w'\|_{L^2_tL^2_x(S_t)}\le B_4\varepsilon(1+t)^{1/20}. \tag{6.11}$$

Here, as before, the L^2_x-norms are taken over $\mathbb{R}^4\backslash\mathcal{K}$, and the weighted $L^2_tL^2_x$ norms are over $S_t=[0,t]\times\mathbb{R}^4\backslash\mathcal{K}$.

In (6.7), we may take $A_0=10C_2$ where C_2 is the constant in (6.5). Clearly, if ε is sufficiently small, then all of these estimates hold for $T=2$ by Theorem 6.1. With this

in mind, we shall then prove that, for $\varepsilon > 0$ smaller than some number depending on $B_1, \dots, B_4$,

(i.) (6.7) is valid with A_0 replaced by $A_0/2$,
(ii.) (6.8)-(6.11) are consequences of (6.7).

It will then follow from the local existence theorem that a solution exists for all $t > 0$ if ε is small enough.

Proof of (i.)*:*

Since v satisfies the better bound (6.5), it suffices to show

$$\sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 1}} \|L^\mu Z^\alpha (w-v)'(t,\,\cdot\,)\|_2^2 + \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|\langle x\rangle^{-3/4} L^\mu Z^\alpha (w-v)'\|^2_{L^2_t L^2_x(S_t)} \le C\varepsilon^4. \tag{6.12}$$

Using the energy integral method as in the proof of (6.5), it follows that the first term on the left side of (6.12) is controlled by

$$C\Bigl(\int_0^t \sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2\, ds\Bigr)^2 + C\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha (w-v)'\|^2_{L^2_t L^2_x(S_t\cap\{|x|<1\})}$$

since $\Box(w-v) = (1-\eta)\Box u$. Thus, by (3.9), the left side of (6.12) is controlled by

$$C\Bigl(\int_0^t \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2\, ds\Bigr)^2 + C\sum_{\substack{|\alpha|+\mu\le 52\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u\|^2_{L^2_t L^2_x(S_t)}. \tag{6.13}$$

We will show that the first term in (6.13) is $O(\varepsilon^4)$. The same techniques can be applied to get the bound for the second term.

We begin by noting that for $|\beta| + \nu \le 53$, $\nu \le 1$, we have

$$\begin{aligned} |L^\nu Z^\beta \Box u(s,y)| \le |u'(s,y)| &\sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} |L^\mu Z^\alpha \partial^2 u(s,y)| \\ &+ \sum_{\substack{|\alpha|+\mu\le 30\\ \mu\le 1}} |L^\mu Z^\alpha u'(s,y)| \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} |L^\mu Z^\alpha u'(s,y)|. \end{aligned} \tag{6.14}$$

Thus, by (4.1) and (4.5), we have

$$
\begin{aligned}
(6.15)\quad & \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2 \\
&\le \frac{C}{(1+s)^{3/2}} \sum_{|\alpha|\le 3} \|Z^\alpha u'(s,\,\cdot\,)\|_2 \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 1}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|>\tilde c s/2)} \\
&\quad + \frac{C}{1+s} \sum_{|\alpha|\le 3} \|\langle y\rangle^{-3/2} Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|<\tilde c s/2)} \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 2}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2 \\
&\quad + \frac{C}{1+s} \sum_{|\alpha|\le 3} \|\langle y\rangle^{-3/2} Z^\alpha u'(s,\,\cdot\,)\|_{L^2(|y|<\tilde c s/2)} \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|\langle s+|y|\rangle L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2 \\
&\quad + C \sum_{|\alpha|\le 3} \|\langle y\rangle^{-3/2} Z^\alpha u'(s,\,\cdot\,)\|_2 \sum_{\mu\le 1} \|L^\mu u'(s,\,\cdot\,)\|_{L^2(|y|<1)} \\
&\quad + C \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|\langle y\rangle^{-3/4} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^2.
\end{aligned}
$$

Here $\tilde c = (1/2)\min_I c_I$. The right side of (6.15) is in turn bounded by

$$
\begin{aligned}
&\le \frac{C}{(1+s)^{3/2}} \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 2}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^2 + C \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|\langle y\rangle^{-3/4} L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^2 \\
&\quad + \frac{C}{(1+s)^2} \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^4 \\
&\quad + C \sum_{|\alpha|\le 3} \|Z^\alpha u'(s,\,\cdot\,)\|_2 \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_{L^2(|y|<\tilde c s/2)}
\end{aligned}
$$

using (4.1) in the third term on the right side of (6.15) if $|y|>\tilde c s/2$. By (6.2) and (6.7), the last term can be absorbed into the left side of (6.15) if ε is small enough. Thus, we see that

$$
\begin{aligned}
(6.16)\quad & \int_0^t \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha \Box u(s,\,\cdot\,)\|_2\, ds \le C\int_0^t \frac{1}{(1+s)^{3/2}} \sum_{\substack{|\alpha|+\mu\le 54\\ \mu\le 2}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^2\, ds \\
&+ C \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|\langle y\rangle^{-3/4} L^\mu Z^\alpha u'\|^2_{L^2_tL^2_x(S_t)} + C\int_0^t \frac{1}{(1+s)^2} \sum_{\substack{|\alpha|+\mu\le 53\\ \mu\le 1}} \|L^\mu Z^\alpha u'(s,\,\cdot\,)\|_2^4\, ds.
\end{aligned}
$$

The first and third terms of (6.16) are easily seen to be $O(\varepsilon^2)$ by (6.2) and (6.10). The second term is also $O(\varepsilon^2)$ by (6.2) and (6.7). This shows that the first term in (6.13) is $O(\varepsilon^4)$ as desired. Since same argument can be used to establish that the second term in (6.13) is also $O(\varepsilon^4)$, we have (6.12) which completes the proof of (*i.*).

Proof of (ii.):

In this section, we complete the proof of Theorem 1.1 by showing that (6.8)-(6.11) are consequences of (6.7). Our first task will be to establish (6.8). Over $|x| \ge \tilde{c}t/2$, the bound follows from (4.1) and (6.7). Over $|x| < \tilde{c}t/2$, we apply (4.6) to bound the left side of (6.8) by

$$C \sum_{\substack{|\alpha|+\mu\le 43\\ \mu\le 1}} \|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2 + C\sum_{|\alpha|\le 42} \|\langle t+r\rangle Z^\alpha \Box w(t,\,\cdot\,)\|_2 + C(1+t)\|w'(t,\,\cdot\,)\|_{L^2(|x|<1)}. \tag{6.17}$$

The first term of (6.17) is $O(\varepsilon)$ by (6.7). When $\Box w$ in the second term is replaced by $\Box u_0 = \eta Q(du, d^2u) + [\Box, \eta]u$, which is supported in $|x| > ct$, this term is seen to be $O(\varepsilon)$ by a Sobolev estimate and (6.1). When $\Box w$ is replaced by $\Box u$ in the second term, we see that it is bounded by

$$C\sum_{|\alpha|\le 43} \|Z^\alpha u'(t,\,\cdot\,)\|_2 \sup_x \Big((1+t+|x|)\sum_{|\alpha|\le 40} |Z^\alpha u'(t,\,\cdot\,)|\Big) \le C\varepsilon^2 + C\varepsilon \sup_x\Big((1+t+|x|)\sum_{|\alpha|\le 40}|Z^\alpha w'(t,\,\cdot\,)|\Big)$$

by (6.2) and (6.7). For ε sufficiently small, the second of these terms can be absorbed into the left side of (6.8). Since $\Box w = \Box u - \Box u_0$, this establishes the desired control on the second term in (6.17).

It remains to bound the last term in (6.17). By the fundamental theorem of calculus, we see that it is dominated by

$$C\int_0^t \sum_{|\alpha|+\mu\le 1} \|L^\mu \partial^\alpha w'(s,\,\cdot\,)\|_{L^2(|x|<1)}\, ds.$$

By (5.3), this is in turn controlled by

$$C\int_0^t \sum_{\substack{|\alpha|+\mu\le 2\\ \mu\le 1}} \|L^\mu\partial^\alpha \Box w(s,\,\cdot\,)\|_2\, ds + C\int_0^t\int \sum_{\substack{|\alpha|+\mu\le 6\\ \mu\le 1}} |L^\mu Z^\alpha \Box w(s,y)|\,\frac{dy\,ds}{|y|^{3/2}}. \tag{6.18}$$

When $\Box w$ is replaced by $\Box u_0$, it follows from (6.1) that both of these terms are $O(\varepsilon)$. In the remaining case, when $\Box w$ is replaced by $\Box u$, it follows from (4.1) that both terms in (6.18) are controlled by

$$C\sum_{\substack{|\alpha|+\mu\le 6\\ \mu\le 1}} \|\langle x\rangle^{-3/4} L^\mu Z^\alpha u'\|^2_{L^2_tL^2_x(S_t)}.$$

This completes the proof of (6.8) since the above is $O(\varepsilon)$ by (6.2) and (6.7).

With (6.8) established, the remainder of the proof follows very similarly to that in [19]. The main exception is how we deal with the boundary term in (2.8). We will only provide a sketch of the arguments that follow exactly as in [19]. The reader may also wish to refer to [20].

Let us begin with the proof of (6.9). In the notation of §2, we have $(\Box_\gamma u)^I = \sum_{\substack{0\le j,k\le 4\\ 1\le J,K\le D}} A^{I,jk}_{JK}\partial_j u^J \partial_k u^K$ and $\gamma^{IJ,jk} = -\sum_{\substack{0\le l\le 4\\ 1\le K\le D}} B^{IJ,jk}_{K,l}\partial_l u^K$. Notice that by (6.2) and (6.8)

$$\|\gamma'(s,\,\cdot\,)\|_\infty \le \frac{C\varepsilon}{(1+s)}. \tag{6.19}$$

In order to prove (6.9), we first estimate the energy of $\partial_t^j u$ for $j\le M\le 100$ using induction on M. By (2.5) and (6.19), we have

$$\partial_t E_M^{1/2}(u)(t) \le C\sum_{j\le M}\|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\|_2 + \frac{C\varepsilon}{1+t}E_M^{1/2}(u)(t). \tag{6.20}$$

Since it follows from elliptic regularity and (6.8) that

$$\begin{aligned}\sum_{j\le M}\|\Box_\gamma \partial_t^j u(t,\,\cdot\,)\|_2 \le \frac{C\varepsilon}{1+t}\sum_{j\le M}\|\partial_t^j u'(t,\,\cdot\,)\|_2 \\ + C\sum_{|\alpha|\le M-41,|\beta|\le M/2}\|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2,\end{aligned}$$

we obtain

$$\partial_t E_M^{1/2}(u)(t) \le \frac{C\varepsilon}{1+t}E_M^{1/2}(u)(t) + C\sum_{|\alpha|\le M-41,|\beta|\le M/2}\|\partial^\alpha u'(t,\,\cdot\,)\partial^\beta u'(t,\,\cdot\,)\|_2 \tag{6.21}$$

since $E_M^{1/2}(u)(t)\approx \sum_{j\le M}\|\partial_t^j u'(t,\,\cdot\,)\|_2$ for ε small.

When $M=40$, the last term in (6.21) drops out. Thus, since (1.4) implies that $E_{100}^{1/2}(u)(0)\le C\varepsilon$, Gronwall's inequality yields

$$\sum_{j\le 40}\|\partial_t^j u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon}. \tag{6.22}$$

For $M>40$, we have to deal with the last term in (6.21). By (4.1), this term is bounded by

$$C\sum_{|\alpha|\le \max(M-38,3+M/2)}\|\langle x\rangle^{-3/4}Z^\alpha u'(t,\,\cdot\,)\|_2^2.$$

Thus, (6.21) and Gronwall's inequality yield,

$$E_M^{1/2}(u)(t)\le C(1+t)^{C\varepsilon}\Big[\varepsilon + \sum_{|\alpha|\le\max(M-38,3+M/2)}\|\langle x\rangle^{-3/4}Z^\alpha u'\|^2_{L^2_tL^2_x(S_t)}\Big]. \tag{6.23}$$

If we use (6.22) and (6.23),

$$E_{100}^{1/2}(u)(t)\le C\varepsilon(1+t)^{C\varepsilon+\sigma} \tag{6.24}$$

would follow for arbitrarily small $\sigma>0$ from a simple induction argument and the following lemma. At every step of the induction, we are using the fact that bounds on $E_M^{1/2}(u)(t)$ yield bounds on $\sum_{|\alpha|\le M}\|\partial^\alpha u'(t,\,\cdot\,)\|_2$ by elliptic regularity.

Lemma 6.2. *Under the above assumptions, if $M \le 100$ and*

$$\sum_{|\alpha|\le M} \|\partial^\alpha u'(t,\cdot)\|_2 + \sum_{|\alpha|\le M-3} \|\langle x\rangle^{-3/4}\partial^\alpha u'\|_{L^2_tL^2_x(S_t)} + \sum_{|\alpha|\le M-4} \|Z^\alpha u'(t,\cdot)\|_2 + \sum_{|\alpha|\le M-6} \|\langle x\rangle^{-3/4}Z^\alpha u'\|_{L^2_tL^2_x(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+\sigma} \tag{6.25}$$

with $\sigma > 0$, then there is a constant C' so that

$$\sum_{|\alpha|\le M-2} \|\langle x\rangle^{-3/4}\partial^\alpha u'\|_{L^2_tL^2_x(S_t)} + \sum_{|\alpha|\le M-3} \|Z^\alpha u'(t,\cdot)\|_2 + \sum_{|\alpha|\le M-5} \|\langle x\rangle^{-3/4}Z^\alpha u'\|_{L^2_tL^2_x(S_t)} \le C'\varepsilon(1+t)^{C'\varepsilon+C'\sigma}. \tag{6.26}$$

Proof of Lemma 6.2: We start by estimating the first term in the left side of (6.26). By (6.2), (6.5), (3.8), and the fact that $\Box(w-v) = (1-\eta)\Box u$, this is dominated by

$$C\varepsilon + C\sum_{|\alpha|\le M-2}\int_0^t \|\partial^\alpha \Box u(s,\cdot)\|_2\,ds + C\sum_{|\alpha|\le M-3}\|\partial^\alpha\Box u\|_{L^2_tL^2_x(S_t)}.$$

If $M \le 40$, we can use (6.2), (6.8), and (6.25) to see that the last two terms are $\le C\varepsilon^2(1+t)^{C\varepsilon+\sigma}$. If $40 < M \le 100$, we can repeat the proof of (6.23) to conclude that the are

$$\begin{aligned}
&\le C\varepsilon^2(1+t)^{C\varepsilon+\sigma} + C\sum_{|\alpha|\le \max(M-40,3+M/2)} \|\langle x\rangle^{-3/4}Z^\alpha u'\|^2_{L^2_tL^2_x(S_t)}\\
&\quad + C\sup_{0\le s\le t}\Big(\sum_{|\alpha|\le M-6}\|Z^\alpha u'(s,\cdot)\|_2\Big)\sum_{|\alpha|\le\max(M-40,3+M/2)}\|\langle x\rangle^{-3/4}Z^\alpha u'\|_{L^2_tL^2_x(S_t)}\\
&\le C\varepsilon^2(1+t)^{2C\varepsilon+2\sigma},
\end{aligned}$$

using the inductive hypothesis (6.25) and the fact that $\max(M-40, 3+M/2) \le M-6$ if $M \ge 40$.

We next establish the bound for the second term in the left of (6.26) using (2.10). With $Y_{M-3,0}(t)$ as in (2.9), it follows as in the proof of (6.25) that

$$\sum_{|\alpha|\le M-3}\|\Box_\gamma Z^\alpha u(t,\cdot)\|_2 \le \frac{C\varepsilon}{1+t}Y^{1/2}_{M-3,0}(t) + C\sum_{|\beta|\le M-40}\|\langle x\rangle^{-3/4}Z^\beta u'(t,\cdot)\|_2^2$$

using (4.1), (6.2), and (6.8). Plugging this into (2.10), we have

$$\partial_t Y_{M-3,0}(t) \le \frac{C\varepsilon}{1+t}Y_{M-3,0}(t) + C\sum_{|\beta|\le M-40}\|\langle x\rangle^{-3/4}Z^\beta u'(t,\cdot)\|_2^2 + C\sum_{|\alpha|\le M-2}\|\langle x\rangle^{-3/4}\partial^\alpha u'(t,\cdot)\|_2^2.$$

By Gronwall's inequality and the fact that $\sum_{|\alpha|\le M-3}\|Z^\alpha u'(t,\cdot)\|_2^2 \le CY_{M-3,0}(t)$ for ε small enough, this yields

$$\sum_{|\alpha|\le M-3}\|Z^\alpha u'(t,\cdot)\|_2^2 \le C(1+t)^{C\varepsilon}\Big(\varepsilon^2 + C\sum_{|\beta|\le M-40}\|\langle x\rangle^{-3/4}Z^\beta u'\|^2_{L^2_tL^2_x(S_t)} + C\sum_{|\alpha|\le M-2}\|\langle x\rangle^{-3/4}\partial^\alpha u'\|^2_{L^2_tL^2_x(S_t)}\Big).$$

The last term is bounded by the right side of (6.26) using the previous step. For the second term in the right, we can apply the inductive hypothesis (6.25) which yields (6.26).

Using (3.9), this in turn implies that the third term in the left of (6.26) satisfies the bounds, which completes the proof. □

This proves (6.24). By elliptic regularity and (6.2), (6.9) follows. It also follows from the lemma that

$$\sum_{|\alpha|\le 98}\|\langle x\rangle^{-3/4}\partial^\alpha w'\|_{L^2_tL^2_x(S_t)} + \sum_{|\alpha|\le 97}\|Z^\alpha w'(t,\cdot)\|_2 + \sum_{|\alpha|\le 95}\|\langle x\rangle^{-3/4}Z^\alpha w'\|_{L^2_tL^2_x(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+\sigma}. \tag{6.27}$$

Here and in what follows σ denotes a small constant that must be taken to be larger at each occurence.

We now proceed to the proof of the estimates involving powers of L. We first estimate $L^\nu\partial^\alpha u'$ in L^2. We then obtain (6.11) and (6.12) for this ν using an inductive argument similar to Lemma 6.2.

The main part of the next step is to show that

$$\sum_{\substack{|\alpha|+\mu\le 92\\ \mu\le 1}}\|L^\mu\partial^\alpha u'(t,\cdot)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}. \tag{6.28}$$

For this, we shall use (2.8). We must first establish (2.7) for $N_0+\nu_0\le 92$, $\nu_0=1$. Arguing as in the proof of (6.23), which uses (4.1), (6.8), and elliptic regularity, we get that for $M\le 92$

$$\begin{aligned}\sum_{\substack{j+\mu\le M\\ \mu\le 1}}\Big(\|\tilde L^\mu\partial_t^j\Box_\gamma u(t,\cdot)\|_2 + \|[\tilde L^\mu\partial_t^j,\Box-\Box_\gamma]u(t,\cdot)\|_2\Big)&\\ \le \frac{C\varepsilon}{1+t}\sum_{\substack{j+\mu\le M\\ \mu\le 1}}\|\tilde L^\mu\partial_t^j u'(t,\cdot)\|_2&\\ + C\sum_{|\alpha|\le M-41}\|\langle x\rangle^{-3/4}L\partial^\alpha u'(t,\cdot)\|_2\sum_{|\alpha|\le 95}\|\langle x\rangle^{-3/4}Z^\alpha u'(t,\cdot)\|_2&\\ + C\sum_{|\alpha|\le \max(M,3+M/2)}\|\langle x\rangle^{-3/4}Z^\alpha u'(t,\cdot)\|_2^2.&\end{aligned}$$

Thus, we obtain (2.7) with $\delta = C\varepsilon$ and

$$H_{1,M-1}(t) = C \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-3/4} L\partial^\alpha u'(t,\,\cdot\,)\|_2^2 + C \sum_{|\alpha|\le 95} \|\langle x\rangle^{-3/4} Z^\alpha u'(t,\,\cdot\,)\|_2^2.$$

Since (1.4) gives $\Big(\int e_0(\tilde{L}^\mu \partial_t^j u)(0,x)\,dx\Big)^{1/2} \le C\varepsilon$ if $\mu + j \le 100$, it follows from (2.8) and (6.27) that for $M \le 92$

$$\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le 1}} \|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma} + C(1+t)^{C\varepsilon} \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-3/4} L\partial^\alpha u'\|^2_{L^2_tL^2_x(S_t)}$$
$$+ C(1+t)^{C\varepsilon} \int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha u'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds. \tag{6.29}$$

By (6.2) and (5.3), this last integral is dominated by $\varepsilon \log(2+t)$ plus

$$\int_0^t \sum_{|\alpha|\le M+1} \|\partial^\alpha w'(s,\,\cdot\,)\|_{L^2(|x|<1)}\,ds \le C\int_0^t \sum_{|\alpha|\le M+2} \|\partial^\alpha \Box w(s,\,\cdot\,)\|_2\,ds$$
$$+ C\int_0^t \int \sum_{|\alpha|\le M+6} |\partial^\alpha \Box w(s,\,\cdot\,)|\,\frac{dy\,ds}{|y|^{3/2}}. \tag{6.30}$$

When w is replaced by u_0, both of these terms are $O(\varepsilon)$ by (6.1). Since $\Box w = \Box u - \Box u_0$, it suffices to consider the case that w is replaced by u. In this case, the right side of (6.30) is controlled by

$$C \sum_{|\alpha|\le 95} \|\langle x\rangle^{-3/4} Z^\alpha u'\|^2_{L^2_tL^2_x(S_t)} + C \sum_{|\alpha|\le 98} \|\langle x\rangle^{-3/4} \partial^\alpha u'\|_{L^2_tL^2_x(S_t)}.$$

Both of these terms are in turn $\le C\varepsilon(1+t)^{C\varepsilon+\sigma}$ by (6.2) and (6.27).

Therefore, by (6.29), we have that

$$\sum_{\substack{|\alpha|+\mu\le M\\ \mu\le 1}} \|L^\mu \partial^\alpha u'(t,\,\cdot\,)\|_2 \le C\varepsilon(1+t)^{C\varepsilon+\sigma}$$
$$+ C(1+t)^{C\varepsilon} \sum_{|\alpha|\le M-41} \|\langle x\rangle^{-3/4} L\partial^\alpha u'\|^2_{L^2(S_t)}.$$

This gives the desired bound when $M \le 40$. Since the analog of Lemma 6.2 is valid when $M = 100$ is replaced by $M = 92$ and u is replaced by Lu, we get (6.28) by a simple induction argument. This same induction also yields, as in the case of no L's,

$$\sum_{\substack{|\alpha|+\mu\le 90\\ \mu\le 1}} \|\langle x\rangle^{-3/4} L^\mu \partial^\alpha w'\|_{L^2_tL^2_x(S_t)} + \sum_{\substack{|\alpha|+\mu\le 89\\ \mu\le 1}} \|L^\mu Z^\alpha w'(t,\,\cdot\,)\|_2$$
$$+ \sum_{\substack{|\alpha|+\mu\le 87\\ \mu\le 1}} \|\langle x\rangle^{-3/4} L^\mu Z^\alpha w'\|_{L^2_tL^2_x(S_t)} \le C\varepsilon(1+t)^{C\varepsilon+C\sigma}. \tag{6.31}$$

Repeating this argument for $L^2 Z^\alpha u'$, it in turn follows from (6.28) and (6.31) that

$$\sum_{\substack{|\alpha|+\mu\le 81\\ \mu\le 2}} \|L^\mu Z^\alpha w'(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\mu\le 79\\ \mu\le 2}} \|\langle x\rangle^{-3/4} L^\mu Z^\alpha w'\|_{L^2_t L^2_x(S_t)} \le C\varepsilon(1+t)^{C\varepsilon + C\sigma},$$

which implies (6.10) and (6.11). This completes the proof of $(ii.)$, and hence the proof of Theorem 1.1.

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